LMI-BASED FRAMEWORK FOR THE SYNTHESIS OF SATURATING CONTROLS LAWS

João M. Gomes da Silva Jr.[†]
jmgomes@delet.ufrgs.br

Cristiane Paim [‡]
cris@lcmi.ufsc.br

Eugênio B. Castelan [‡]
eugenio@lcmi.ufsc.br

[†] UFRGS - Departamento de Engenharia Elétrica
Av. Osvaldo Aranha 103, 90035-190 Porto Alegre, RS, BRAZIL

[‡] UFSC - Departamento de Automação e Sistemas
P.O. Box 476, 88040-900 Florianópolis, SC, BRAZIL

ABSTRACT – Considering linear open-loop systems, the design of full state feedback and observer-based output feedback control laws subject to control input saturation are addressed. The local asymptotic stability of the nonlinear closed-loop system is studied through a quadratic approach. In this context, conditions guaranteeing the local asymptotic stability of the nonlinear (saturated) closed-loop system are stated. From these conditions, a framework based on Linear Matrix Inequalities (LMIs) is proposed to compute control laws in order to ensure the asymptotic stability of a given admissible set of initial states and a certain degree of time-domain performance for the closed-loop system in a neighborhood of the origin.

1 INTRODUCTION

In the last years, the problem of the stabilization of linear systems subject to control saturation has received the attention of many authors (see for example (Bernstein and Michel, 1995)). The interest in this problem is mainly motivated by the fact that the negligence of the control bounds can be source of limit cycles, parasitic equilibrium points and even of the instability of the closed-loop system. The works found in the literature can be classified in three contexts of closed-loop stability, namely the global, the semi-global and the local stability.

It is well-known (Burgat and Tarbouriech, 1996), (Sussmann et al., 1994) that the global stabilization of linear systems subject to control saturation can be achieved only when the open-loop system is not strictly unstable, i.e., in the continuous-time case, it must have all its poles in the left half complex plane. However, the physical interest of the global stability is questionable since, in general, the system is restricted to operate in a limited zone of the state space. Hence, in our point of view, the semi-global stabilization (Lin and Saberi, 1993),(Alvarez-Ramirez et al., 1994) and the local stabilization (Gutman and Hagander, 1985),(Burgat and Tarbouriech, 1996) represent more realistic approaches. The objective is to guarantee the asymptotic stability not of the whole state space but only of a given set $\mathcal{X}_0$ of admissible initial states, that can be viewed as the zone of operation of the system. In this sense, two approaches can be cited: the linear, where the saturation is avoided, and the nonlinear, where the effective occurrence of saturation and the nonlinear behavior of the closed-loop system are taken into account. The design by the first approach can be accomplished by computing the control law in order to guarantee the positive invariance of a set $\mathcal{S} \supset \mathcal{X}_0$ such that $\mathcal{S}$ is contained in the region of linear behavior of the closed-loop system. Concerning this approach, straightforward solutions based on LMI conditions and the determination of quadratic Lyapunov domains contained in the region of linear behavior and containing the region $\mathcal{X}_0$ can be found in the literature (see (Boyd et al., 1994), for example). However, if the set $\mathcal{X}_0$ is relatively large and the closed-loop performance specifications are very stringent, it may not exist a solution to the problem. On the other hand, we can say that the proposition of methods that take into account the effective saturation and that conciliates performance, robustness and the stabilization of a large set of initial conditions (basin of attraction) for the closed loop system remains a challenge. Hence, this paper is concerned with the second approach.

Given a set of admissible initial conditions $\mathcal{X}_0$ to be stabilized, our objective is to compute a saturating state feedback or an observer-based output feedback control law that guarantees both the asymptotic convergence to the origin of all trajectories emanating from $\mathcal{X}_0$ and a certain degree of time-domain performance for the closed-loop system in a neighborhood of the origin. In this aim, we use a local representation of the saturated system deduced from the differential inclusions theory. This representation consists in a polytopic model valid in a certain polyhedral set in the state space. Based on this model, some conditions expressed as linear matrix inequalities (LMIs) and biaffine matrix inequalities (BMIs) are stated for determining a state feedback and state-observer matrices in order to verify both stabilization and performance requirements. Since the numerical solution of BMIs is a difficult task to accomplish, an LMI-framework, based on some relaxation schemes combined with an optimization problem, is proposed to handle the problem. The results are presented in two parts: the state feedback case and the observer-based output feedback. A numerical example is provided to illustrate the application of the proposed algorithms.

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Notations. For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \ldots, n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A^T denotes the transpose of A. **Co** denotes a convex hull.

2 STATE FEEDBACK SYNTHESIS

Consider a linear continuous-time system defined by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are respectively the state vector and the control vector. Matrices A and B are real constant matrices of appropriate dimensions and the pair (A, B) is supposed to be stabilizable.

The control vector is subject to linear constraints which define the polyhedral compact region $\Omega \subset \Re^m$:

$$\Omega \stackrel{\triangle}{=} \{u \in \Re^m \; ; \; -\rho \preceq u \preceq \rho\} \quad \text{with} \quad \rho \succ 0.$$

Consider the following saturated state feedback control law

$$u(t) = sat(Fx(t)) \qquad F \in \Re^{m \times n},$$

where each component is defined, $\forall\, i = 1, \ldots, m$, by

$$\begin{aligned} u_{(i)} &= (sat(Fx(t)))_{(i)} \\ &= \begin{cases} -\rho_{(i)} & \text{if } F_{(i)}x(t) < -\rho_{(i)} \\ F_{(i)}x(t) & \text{if } -\rho_{(i)} \leq F_{(i)}x(t) \leq \rho_{(i)} \\ \rho_{(i)} & \text{if } F_{(i)}x(t) > \rho_{(i)} \end{cases} \end{aligned} \tag{2}$$

Thus, the closed-loop system is given by the following *nonlinear* model :

$$\dot{x}(t) = Ax(t) + Bsat(Fx(t)) \tag{3}$$

It is worth noticing that inside the domain $S(F, \rho)$ defined as

$$S(F, \rho) \stackrel{\triangle}{=} \{x \in \Re^n \; ; \; -\rho \preceq Fx \preceq \rho\} \tag{4}$$

the control inputs do not saturate and therefore the evolution of the closed-loop system is described by the *linear* model

$$\dot{x}(t) = (A + BF)x(t) \tag{5}$$

Outside of $S(F, \rho)$, the control inputs saturate and the stability of the system must be analyzed by considering equation (3).

Considering a set of admissible initial conditions (that can represent a safe operation zone for the closed-loop system or a zone where the state of the system can be driven by the action of nonpersistent disturbances), $\mathcal{X}_0$, the problem to be solved in the sequel is stated as follows:

Problem 1 *Consider system (1). Compute a saturated state feedback control law defined by (2), such that for all initial states belonging to $\mathcal{X}_0$ the corresponding trajectories converge asymptotically to the origin. In addition, this control law should also guarantee a certain time-domain performance specification inside the domain of linearity defined by (4).*

This problem can be interpreted as a problem of *Local Asymptotic Stabilization*. It can be solved if we can compute a state feedback that ensures the local stability of system (3) in a region containing the set $\mathcal{X}_0$. In particular, a solution can be found if we are able to compute a matrix F that makes a set $\mathcal{S} \supset \mathcal{X}_0$ positively invariant and contractive w.r.t the saturated system (3). In this paper we deal with a quadratic approach for synthesis and thus we are particularly interested in ellipsoidal domains of invariance and contractivity.

2.1 Polytopic Representation

In order to state the main results of the paper, we define an appropriate representation for the saturated system. The basic idea is to represent the saturated system by a polytopic model. This kind of representation was first introduced in (Molchanov and Pyatnitskii, 1989) and has been applied in the specific case of system (3) in (Burgat and Tarbouriech, 1996), (Gomes da Silva Jr. et al., 1997) and (Gomes da Silva Jr. and Tarbouriech, 1999c).

Note that the ith entry of the saturated control law defined in (2) can also be written as

$$(sat(Fx(t)))_{(i)} = \alpha(x(t))_{(i)} F_{(i)} x(t) \tag{6}$$

with

$$\alpha(x(t))_{(i)} \stackrel{\triangle}{=} \begin{cases} \frac{-\rho_{(i)}}{F_{(i)}x(t)} & \text{if } F_{(i)}x(t) < -\rho_{(i)} \\ 1 & \text{if } -\rho_{(i)} \leq F_{(i)}x(t) \leq \rho_{(i)} \\ \frac{\rho_{(i)}}{F_{(i)}x(t)} & \text{if } F_{(i)}x(t) > \rho_{(i)} \end{cases} \tag{7}$$

The coefficient $\alpha(x(t))_{(i)}$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, smaller is $\alpha(x(t))_{(i)}$, farther is the state vector from the region of linearity (4). Notice that $\alpha(x(t))_{(i)}$ is a function of $x(t)$. For the sake of simplicity, in the sequel we denote $\alpha(x(t))_{(i)}$ as $\alpha(t)_{(i)}$.

Define from the vector $\alpha(t) \in \Re^m$ a diagonal matrix $D(\alpha(t)) \stackrel{\triangle}{=} diag(\alpha(t))$. Thus, system (3) can be rewritten as

$$\dot{x}(t) = (A + BD(\alpha(t))F)x(t) \tag{8}$$

Now, let $0 < \underline{\alpha}_{(i)} \leq 1$ be a lower bound to $\alpha(t)_{(i)}$ and define the vector $\underline{\alpha} \stackrel{\triangle}{=} [\underline{\alpha}_{(1)}, ..., \underline{\alpha}_{(m)}]^T$. The vector $\underline{\alpha}$ is associated to the following region in the state space:

$$S(F, \rho^\alpha) = \{x \in \Re^n \; ; \; -\rho^\alpha \preceq Fx \preceq \rho^\alpha\} \tag{9}$$

where $\rho^\alpha_{(i)} \stackrel{\triangle}{=} \frac{\rho_{(i)}}{\underline{\alpha}_{(i)}}, \forall i = 1, \ldots, m$.

Consider now all the possible m-order vectors such that the ith entry takes the value 1 or $\underline{\alpha}_{(i)}$. Hence, there exists a total of 2^m different vectors. By denoting each one of these vectors by γ_j, $j = 1, \ldots, 2^m$, define the following matrices: $D_j(\underline{\alpha}) = D(\gamma_j) = diag(\gamma_j)$ and $A_j = A + BD_j(\underline{\alpha})F$. Note that the matrices A_j are the vertices of a convex polytope of matrices. If $x(t) \in S(F, \rho^\alpha)$ it follows that $(A + BD(\alpha(t))F) \in \mathbf{Co}\{A_1, A_2, \ldots, A_{2^m}\}$. Hence, if $x(t) \in S(F, \rho^\alpha)$, $\dot{x}(t)$ can be determined from an appropriate convex linear combination of matrices A_j at time t, that is:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t)) A_j x(t) \tag{10}$$

with $\sum_{j=1}^{2^m} \lambda_j(x(t)) = 1, \;\; \lambda_j(x(t)) \geq 0$.

It should be pointed out that model (10) represents the saturated system only in $S(F, \rho^\alpha)$. Actually, if $x(t) \in S(F, \rho^\alpha)$, the polytopic model (10) can be used to determine $\dot{x}(t)$.

2.2 Main Results

In order to solve Problem 1 we should compute a state feedback that guarantees the local stability of system (3) in a region that contains the set $\mathcal{X}_0$. Furthermore, when this system operates in the region of linearity, i.e., the closed-loop system is described by (5), a certain degree of time-domain performance should be guaranteed. This kind of specification can, in general, be achieved by placing the poles of $(A+BF)$ in a suitable region of the half left complex plane (Chiali and Gahinet, 1996).

We suppose then that the following data is given:

• A vector ρ of control bounds.

• A set of initial conditions $\mathcal{X}_0$ defined by a polyhedral set described by its vertices:

$$\mathcal{X}_0 \triangleq \mathbf{Co}\{v_1, \ldots, v_{n_v}\}, \;\; v_s \in \Re^n \;\; \forall s = 1, \ldots, n_v \qquad (11)$$

• A region $\mathcal{D}$, contained in the left half complex plane, defined as follows (Chiali and Gahinet, 1996):

$$\mathcal{D} \triangleq \{s \in \mathcal{C} \;\; ; \;\; (H + sQ + \bar{s}Q^T) < 0\} \qquad (12)$$

where H and Q are $l \times l$ symmetric real matrices and s is a complex number with its conjugate $\bar{s}$. We assume that the time-domain requirements in the region of linear behavior of are satisfied if the poles $(A + BF)$ are located in a region $\mathcal{D}$.

Hence, considering the data above and the polytopic representation of the saturated system, if we are able to find a matrix F, a vector $\underline{\alpha}$ and a set $\mathcal{E}$ in the state space such that

a) the set $\mathcal{E}$ is contractive with respect to the trajectories of he differential inclusion (10), and

b) $\mathcal{X}_0 \subset \mathcal{E} \subset S(F, \rho^\alpha)$,

then we can conclude that all the trajectories of the saturated system (3) starting in $\mathcal{E}$ (and, in consequence, all the trajectories starting in $\mathcal{X}_0$) converge asymptotically to the origin. In this case, the set $\mathcal{E}$ is a *domain of asymptotic stability* for the system (3). If, in addition, the poles of $(A + BF)$ are contained in $\mathcal{D}$, Problem 1 is solved. These ideas are formalized in the following proposition.

Proposition 1 *If there exist matrices $W = W^T > 0$, $W \in \Re^{n\times n}$, and $Y \in \Re^{m\times n}$ and a vector $\underline{\alpha} \in \Re^m$, satisfying the following matrix inequalities:*

$$\begin{array}{ll}
(i) & WA^T + AW + BD_j(\underline{\alpha})Y + Y^T D_j(\underline{\alpha})^T B^T < 0 \\
 & \forall j = 1, \ldots, 2^m \\
(ii) & \begin{bmatrix} W & Y_{(i)}^T \\ Y_{(i)} & (\rho_{(i)}/\underline{\alpha}_{(i)})^2 \end{bmatrix} \geq 0 \;\; , \;\; \forall i = 1, \ldots, m \\
(iii) & \begin{bmatrix} 1 & v_s^T \\ v_s & W \end{bmatrix} \geq 0 \;\; , \;\; \forall s = 1, \ldots, n_v \\
(iv) & 0 < \underline{\alpha}_{(i)} \leq 1, \; i = 1, \ldots, m \\
(v) & [h_{ij}W + q_{ij}(AW + BY) + q_{ij}(AW + BY)^T] < 0 \\
 & 1 \leq i, j \leq l
\end{array}$$

then $F \triangleq YW^{-1}$ solves Problem 1 and the set $\mathcal{E} \triangleq \{x \in \Re^n \; ; \; x^T P x \leq 1\}$, with $P = W^{-1}$, is a domain of asymptotic stability for system (3).

Proof : If there exist matrices $W = W^T > 0$ and Y, and a vector $\underline{\alpha}$ satisfying the matrix inequalities (i) to (v) it follows that:

• From inequalities (i) one obtains

$$\sum_{j=1}^{2^m} \lambda_j(t)(WA^T + AW + BD_j(\underline{\alpha})Y + Y^T D_j(\underline{\alpha})^T) < 0 \quad (13)$$

with $\sum_{j=1}^{2^m} \lambda_j(t) = 1$, $\lambda_j(t) \geq 0$.

• Inequality (ii) ensures that ellipsoid $\mathcal{E}$ is contained in the region $S(F, \rho^\alpha)$ with $F = YW^{-1}$ (Gomes da Silva Jr. et al., 1997).

• LMIs (iii) guarantee that $\mathcal{X}_0$ defined by (11) is contained in the ellipsoid $\mathcal{E}$ (Boyd et al., 1994).

• LMI (v) guarantees that all the eigenvalues of $(A + BF)$ are contained in region $\mathcal{D}$ (Chiali and Gahinet, 1996).

Suppose now that $x(t) \in \mathcal{E}$. Since $\mathcal{E} \subset S(F, \rho^\alpha)$ and (iv) holds, $\dot{x}(t)$ can be computed from the polytopic model (10) with appropriate $\lambda_j(t)$, $j = 1, \ldots, 2^m$ and matrices A_j defined from the coefficients of saturation $\underline{\alpha}_{(i)}$ and $F = YW^{-1}$. Pre and post multiplying (13) by P, it follows that:

$$x^T \left[\left(\sum_{j=1}^{2^m} \lambda_j(t) A_j \right)^T P + P \left(\sum_{j=1}^{2^m} \lambda_j(t) A_j \right) \right] x < 0$$

$$\dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) < 0$$

Since this reasoning is valid $\forall x(t) \in \mathcal{E}$, $x(t) \neq 0$, we can conclude that $\mathcal{V}(x(t)) \triangleq x(t)^T P x(t)$ is a local strictly decreasing Lyapunov function for the saturated system (3) in $\mathcal{E}$ and thus the ellipsoid $\mathcal{E}$ is a contractive domain w.r.t system (3). Since $\mathcal{X}_0 \subset \mathcal{E}$, the asymptotic convergence to the origin of all trajectories of system (3) emanating from $\mathcal{X}_0$ is guaranteed. The LMI (i) guarantees the performance in the region of linearity $S(F, \rho)$. $\diamondsuit$

2.3 LMI framework

The variables to be found by applying Proposition 1 are W, Y and $\underline{\alpha}$. However, note that inequalities (i) and (ii) of Proposition 1 are bilinear (BMI) in decision variables Y and $\underline{\alpha}$, whereas relations (iii), (iv), and (v) are linear (LMI) in W, Y and $\underline{\alpha}$.

An easy and straightforward way to overcome the problem of solving the BMIs is to fix, a priori, the value of the components of $\underline{\alpha}$ (Gomes da Silva Jr. et al., 1997). In this case, inequalities (i) and (ii) become LMIs and, given $(\rho, \mathcal{X}_0, \mathcal{D})$, it is possible to solve constraints $(i) - (v)$ of Proposition 1, as a feasibility problem, with efficient numeric algorithms (Boyd et al., 1994). Of course, considering a fixed vector $\underline{\alpha}$ and the given data, it may actually be impossible to find a feasible solution. In fact, considering an scaling factor β, $\beta > 0$, the maximum homothetic set to $\mathcal{X}_0$, $\beta\mathcal{X}_0$, that can be stabilized considering the fixed $\underline{\alpha}$, can be obtained solving the following convex optimization problem with LMI constraints:

$$\max_{\beta,W,Y} \beta$$
subject to
$$\begin{cases} \begin{bmatrix} 1 & \beta v_i^T \\ \beta v_i & W \end{bmatrix} > 0 \ , \ \forall i = 1,\ldots,n_v \\ \\ \text{LMIs } (i), (ii) \text{ and } (v) \text{ of Proposition 1} \end{cases} \quad (14)$$

Hence, if the optimal value of β, $\beta^\star$, is greater or equal to 1, it means that it is possible to find a solution considering the fixed $\underline{\alpha}$ and the given data $(\rho, \mathcal{X}_0, \mathcal{D})$. We conjecture that smaller are the components of vector $\underline{\alpha}$, greater should be the optimal value of the scalar β, that is, it is possible to stabilize larger domains of admissible initial states (see the numerical example in section 4). Note that the idea is to render the problem less conservative by allowing more control saturation. Hence, for a given region $\mathcal{D}$ and a region $\mathcal{X}_0$, we can consider an iterative scheme where we decrease the components of $\underline{\alpha}$ in each iteration until to find an optimal solution $(W^\star, Y^\star, \beta^\star)$ for (14) with $\beta^\star \geq 1$. In this case two issues arise: how to choose the initial vector $\underline{\alpha}$ and how exactly to decrease the components of $\underline{\alpha}$ (if $\beta^\star < 1$). These issues can be considered as open problems and one simple way of handling them is to apply trial and error procedures. In particular, for mono-input systems, since $\underline{\alpha}$ is a scalar, the optimal solution of problem (14) can be searched over a grid on this scalar. This strategy can be pursued even in the case $m = 2$, where a bidimensional grid must be considered.

On the other hand, we can try to solve (14) directly by considering the problem with BMI constraints. However, as pointed in (Goh et al., 1996), the methods proposed in the literature for solving BMIs present their worst-case complexities exponential and therefore the required computational effort may be unreasonably large. Moreover, BMI-based problems are not convex and so we cannot guarantee that the obtained solution is a global optimum. In order to overcome this computational difficulty, we can approximate the solution of BMI optimization problems via polynomial-time algorithms, for example by using schemes based on LMI relaxations (LMIR). With this aim we propose the following 2-step iterative algorithm:

Algorithm 1 :

Step 1 : *Given $\underline{\alpha}$, solve (14) for W, Y and β (LMIR 1).*

Step 2 : *Given Y, solve (14) for W, $\underline{\alpha}$ and β (LMIR 2).*

The iteration between these two steps stops when a desired precision for β is achieved. If $\beta^\star \geq 1$, it is possible to stabilize the system (3) for all initial conditions in $\mathcal{X}_0$ by considering the pole placement of $(A + BF)$ inside $\mathcal{D}$. In particular, all intermediate solutions with $\beta > 1$ are solutions to Problem 1. Hence, this kind of approach solves, in part, the problem of the choice of vector $\underline{\alpha}$ by using robust and available packages to solve LMIs (Gahinet et al., 1995)

Remark 1 *It is worth to be noticed that if we start the algorithm with $\underline{\alpha} = 1_m$, the convergence to a solution ($\beta^\star$,$W^\star$,$Y^\star$,$\underline{\alpha}^\star$) is ensured provided that the pair (A, B) is controllable. This follows from the fact that an optimal solution for LMIR 1 is also a feasible solution for LMIR 2 and vice-versa. Of course, taking different initial vectors $\underline{\alpha}$ the proposed algorithm can converge to different values of ($\beta^\star$,$W^\star$,$Y^\star$,$\underline{\alpha}^\star$).*

Remark 2 *The result of Proposition 1 can be applied to stable or unstable open-loop systems. However, we should take into account that Proposition 1 furnishes only a sufficient condition to solve Problem 1 by considering the data $(\rho, \mathcal{X}_0, \mathcal{D})$. Note also that, when the open-loop system is unstable, a necessary condition for the existence of a solution for the problem is that the set $\mathcal{X}_0$ must be contained in the controllable region of the system (1) with constrained controls.*

3 FULL STATE OBSERVER CASE

Consider a full-order state observer for system (3) given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - LC(x(t) - \hat{x}(t)) \quad (15)$$

where $\hat{x}(t) \in \Re^n$ is the estimate of the state and $L \in \Re^{n\times q}$ define the estimation dynamics. The applied control is now given by:

$$u(t) = sat(F\hat{x}(t)) \quad (16)$$

where $sat(F\hat{x}(t))$ is defined analogously to (2).

Problem 2 *Consider system (1). Compute an observer-based output feedback control law defined by (15) and (16), such that for all the initial states belonging to $\mathcal{X}_0$ the corresponding trajectories converge asymptotically to the origin. In addition, this control law should also guarantee a certain time-domain performance specification inside the domain of linearity of the system.*

In order to state some conditions to solve Problem 2, we shall consider the representation of the augmented closed-loop system (system + observer) in a particular basis of the state space. Let then $e = x - \hat{x}$ be the estimate error and consider the following similarity transform:

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (17)$$

In the new basis, the augmented closed-loop system is given by (for the sake of simplicity we do not consider the time dependence explicitly):

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0_n \\ 0_n & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad (18)$$

$$u = sat(\begin{bmatrix} F & -F \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}) \quad (19)$$

Proposition 2 *Consider that the initial state of the observer is equal to zero ($\hat{x}(0) = 0$). If there exist matrices $P_1 = P_1^T > 0$, $P_1 \in \Re^{n\times n}$, $P_2 = P_2^T > 0$,$P_2 \in \Re^{n\times n}$, $U \in \Re^{n\times q}$, $F \in \Re^{m\times n}$ and a vector $\underline{\alpha} \in \Re^m$, satisfying the following matrix inequalities:*

$$(i) \quad \begin{bmatrix} A_j^T P_1 + P_1 A_j & -P_1 B_j F \\ -(P_1 B_j F)^T & A^T P_2 + C^T U^T + P_2 A + UC \end{bmatrix} < 0, \quad j = 1,\ldots,2^m$$

$$(ii) \quad \begin{bmatrix} P_1 & 0_n & \underline{\alpha}_{(i)} F_{(i)}^T \\ 0_n & P_2 & -\underline{\alpha}_{(i)} F_{(i)}^T \\ \underline{\alpha}_{(i)} F_{(i)} & -\underline{\alpha}_{(i)} F_{(i)} & \rho_{(i)}^2 \end{bmatrix} \geq 0 \quad i = 1,\ldots,m$$

$$(iii) \quad \begin{bmatrix} v_s^T & v_s^T \end{bmatrix} \begin{bmatrix} P_1 & 0_n \\ 0_n & P_2 \end{bmatrix} \begin{bmatrix} v_s \\ v_s \end{bmatrix} \leq 1 \quad s = 1,\ldots,n_s$$

$$(iv) \quad 0 < \underline{\alpha}_{(i)} \leq 1, \ i = 1,\ldots,m$$

$$(v) \quad \begin{cases} h_{ij} P_1 + q_{ij} P_1 (A + BF) + q_{ij} P_1 (A + BF)^T < 0 \\ h_{ij} P_2 + q_{ij} P_2 (A + UL) + q_{ij} P_2 (A + UL)^T < 0 \\ 1 \leq i,j \leq l \end{cases}$$

where $A_j = A + BD_j(\underline{\alpha})F$ and $B_j = BD_j(\underline{\alpha})$, then the observer-based output feedback control law defined by (15)-(16), with $L = P_2^{-1}U$ and F, solves Problem 2.

Proof : Take into account the definition of matrices $D(\underline{\alpha})$ given in section 2.1, the behavior of the closed loop system (18)-(19) in the set $S(\mathcal{F}, \rho^\alpha) \triangleq \{\tilde{x} \in \Re^{2n} \ ; \ -\rho^\alpha \preceq \mathcal{F}\tilde{x} \preceq \rho^\alpha\}$, with $\mathcal{F} \triangleq \begin{bmatrix} F & -F \end{bmatrix}$ and $\tilde{x} \triangleq \begin{bmatrix} x \\ e \end{bmatrix}$, can be represented by the following polytopic differential inclusion:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \sum_{j=1}^{2^m} \lambda_j \begin{bmatrix} A + BD_j(\underline{\alpha})F & -BD_j(\underline{\alpha})F \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad (20)$$

with $\sum_{j=1}^{2^m} \lambda_j = 1, \lambda_j \geq 0$.

Consider now the following definitions:

$$\mathcal{P} \triangleq \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$$\mathcal{A}_j \triangleq \begin{bmatrix} A + BD_j(\underline{\alpha})F & -BD_j(\underline{\alpha})F \\ 0 & A + LC \end{bmatrix}$$

$$\mathcal{V}(\tilde{x}) \triangleq \tilde{x}^T \mathcal{P} \tilde{x} \ ; \ \tilde{\mathcal{E}} \triangleq \{\tilde{x} \in \Re^{2n} \ ; \ \tilde{x}^T \mathcal{P} \tilde{x} \leq 1\}$$

$$\tilde{\mathcal{X}}_0 \triangleq \mathbf{Co}\{\tilde{v}_s\}, \ \ \tilde{v}_s \triangleq [v_s^T \ \ v_s^T]^T, \ \ s = 1, \ldots, n_v$$

Then it follows that:

- (i) and (iv) guarantees that

$$\tilde{x}^T (\sum_{j=1}^{2^m} \mathcal{P}\mathcal{A}_j + \sum_{j=1}^{2^m} \mathcal{A}_j^T \mathcal{P}) \tilde{x} < 0$$

for every $\tilde{x} \in S(\mathcal{F}, \rho^\alpha)$.

- Considering that $\hat{x}(0) = 0$, for $x(0) = v_s$ it follows that $\tilde{x}(0) = [v_s^T \ \ v_s^T]^T$. Hence, (ii) and (iii) ensures that

$$\tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{E}} \subset S(\mathcal{F}, \rho^\alpha)$$

- (v) ensures that the eigenvalues of the linear closed-loop system (not saturated) are located in the region $\mathcal{D}$, which corresponds to the satisfaction of the time-domain performance requirement.

Suppose now that $\tilde{x} \in \tilde{\mathcal{E}}$. Since $\tilde{\mathcal{E}} \subset S(\mathcal{F}, \rho^\alpha)$ it follows that $\dot{\tilde{x}}$ can be computed by (20) and

$$\tilde{x}^T \mathcal{P} \dot{\tilde{x}} + \dot{\tilde{x}}^T \mathcal{P} \tilde{x} < 0$$

Since this reasoning is valid $\forall \tilde{x}(t) \in \tilde{\mathcal{E}}$, $\tilde{x}(t) \neq 0$, we can conclude that $\tilde{\mathcal{V}}(x(t))$ is a local strictly decreasing Lyapunov function for the system (18)-(19) in $\tilde{\mathcal{E}}$ and thus this set is a contractive domain. From the assumption that $\hat{x}(0) = 0$, it follows that $x(0) \in \mathcal{X}_0$ implies that $\tilde{x}(0) \in \tilde{\mathcal{X}}_0$. Since $\tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{E}}$ we can conclude that $\forall x(0) \in \mathcal{X}_0$ the asymptotic convergence of the trajectories to the origin is ensured. The LMI (v) guarantees the performance in the region of linearity, which concludes the proof. $\diamondsuit$

3.1 LMI Framework

Matrix inequalities of Proposition 2 are non-linear and non-convex with respect to variables P_1, P_2, U, F and $\underline{\alpha}$. However, for a given matrix F and a given vector $\underline{\alpha}$ they become LMIs.

Hence, suppose that a solution $(F, \underline{\alpha})$ to Problem 1 has been obtained from Algorithm 1. If the states are not available for measurement, the idea is to construct an state observer and apply the computed matrix F to the estimated state. In this case, the observer should be designed in order to keep the local stabilization in $\mathcal{X}_0$ (this fact can be seen as an analogy to the *Loop Transfer Recovery problem* when robustness issues are considered). With this aim, based on the condition established in Proposition 2, we propose the following algorithm to compute the observer (i.e. matrix L):

Algorithm 2 :

Step 1: *Compute $(F, \underline{\alpha})$ by Algorithm 1*

Step 2: *Solve the optimization problem:*

$$\min_{\eta, P_1, P_2, U} \ \eta$$

subject to

$$\begin{bmatrix} v_s^T & v_s^T \end{bmatrix} \begin{bmatrix} P_1 & 0_n \\ 0_n & P_2 \end{bmatrix} \begin{bmatrix} v_s \\ v_s \end{bmatrix} \leq \eta \quad s = 1, \ldots, n_s \qquad (21)$$

LMIs $(i), (ii)$ and (v) of Proposition 2

Note that the minimization of η is equivalent to the maximization of scaling coefficient β by considering $\eta = \beta^{-2}$. Hence, if $\eta^{-1/2} = \beta \geq 1$, the Problem 2 has a solution. Otherwise, it is not possible to find a solution from the proposed method considering the given F and $\underline{\alpha}$.

Remark 3 *In the proposed approach, we consider a particular class of matrix $\mathcal{P}$ under a block diagonal form. Hence, if the obtained η is such that $\beta \leq 1$, before conclude that there is no solution with the proposed method, we can try to verify if there exists another ellipsoidal stability region (associated to a full matrix $\mathcal{P}$) for system (18)-(19), that contains the set $\tilde{\mathcal{X}}_0$. This can be accomplished by applying the results proposed in (Gomes da Silva Jr. and Tarbouriech, 1999c).*

4 NUMERICAL EXAMPLE

The numerical results presented in this section were obtained by using the MATLAB LMI Control Toolbox (Gahinet et al., 1995).

Consider the control of two inverted pendulums in cascade where the system matrices are given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.8 & 0 & -9.8 & 0 \\ 0 & 0 & 0 & 1 \\ -9.8 & 0 & 2.94 & 0 \end{bmatrix} \ ; \ B = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 0 \\ -2 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Notice that matrix A is unstable (the eigenvalues of A are: $4.0930, -4.0930, \pm 2.0032i$).

Consider that the control bounds are defined by (1) with $\rho = [10 \;\; 10]^T$. Suppose we want to stabilize the following set of admissible initial conditions:

$$\mathcal{X}_0 = \mathbf{Co}\{ \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0 \\ -0.5 \\ 0 \end{bmatrix} \}$$

The region $\mathcal{D}$ we consider corresponds to a δ-shift, that is, $\mathcal{D} = \{s \in \mathcal{C} \; ; \; \Re\{s\} \leq -\delta \; ; \; \delta > 0\}$. Hence greater is δ, greater tends to be the rate of the convergence of the trajectories to the origin.

Considering the data above, table 1 shows the final values of $\underline{\alpha}$ and β obtained from the iterative algorithm proposed in section 2.3 from different initial vectors $\underline{\alpha}$ and scalars δ. $\beta_{initial}$ and β_{final} denote the optimal value of β obtained by the iterative algorithm from respectively $\underline{\alpha}_{initial}$ and $\underline{\alpha}_{final}$.

Regarding table 1 we can notice the following: smaller are the components of $\underline{\alpha}$, greater is the β obtained from (14). This illustrates the fact that by allowing saturation we can stabilize the system for a larger set of initial conditions. Besides, more stringent is the performance requirement (larger δ, in this case) smaller is the region of admissible initial states for which we can find a solution.

δ	$\underline{\alpha}_{initial}$	$\beta_{initial}$	$\underline{\alpha}_{final}$	β_{final}
1	$[1.00 \;\; 1.00]^T$	1.9883	$[0.7912 \;\; 0.8950]^T$	2.3112
1	$[0.50 \;\; 0.50]^T$	2.3652	$[0.4999 \;\; 0.5000]^T$	2.3653
1	$[0.25 \;\; 0.25]^T$	2.5607	$[0.2495 \;\; 0.2500]^T$	2.5613
2	$[1.00 \;\; 1.00]^T$	0.9731	$[0.7232 \;\; 0.8430]^T$	1.2154
2	$[0.50 \;\; 0.50]^T$	1.2930	$[0.4997 \;\; 0.5000]^T$	1.2931
2	$[0.25 \;\; 0.25]^T$	1.5378	$[0.2493 \;\; 0.2500]^T$	1.5388
3	$[1.00 \;\; 1.00]^T$	0.5707	$[0.6850 \;\; 0.8039]^T$	0.7519
3	$[0.50 \;\; 0.50]^T$	0.8302	$[0.5000 \;\; 0.5000]^T$	0.8316
3	$[0.25 \;\; 0.25]^T$	1.0737	$[0.2485 \;\; 0.2500]^T$	1.0747

Table 1: Algorithm performance

Comparing the solution obtained by avoiding saturation (i.e. $\underline{\alpha} = [1 \;\; 1]$) with the solution take into account the nonlinear behavior of the system, we can observe that:

- for $\delta = 1$, it is possible to obtain a solution for a set of admissible initial states 28% larger.
- for $\delta = 2$, it is possible to obtain a solution for a set of admissible initial states 58% larger.
- for $\delta = 3$, it is possible to obtain a solution for a set of admissible initial states 88% larger.

Now, in order to illustrate the determination of an observer-based control law, consider the following matrix $F = YW^{-1}$ that verifies the conditions of Proposition 1 considering the set $\mathcal{X}_0$ given above, $\underline{\alpha} = [0.7912 \;\; 0.8950]^T$ and $\delta = 1$:

$$F = \begin{bmatrix} -5.5317 & -1.8986 & 4.3732 & 0.2973 \\ 3.8029 & 1.1964 & -3.3488 & -0.4414 \end{bmatrix}$$

A matrix L that solves Problem 2 is computed by applying the step 2 of the algorithm 2. To prevent high estimation gains, an additional constraint was used in order to bound the closed-loop observer poles to a strip defined by -1 and -50. We obtained

$$L = \begin{bmatrix} -47.0332 & 5.0541 \\ -62.6793 & 13.0173 \\ 0.2415 & -45.4971 \\ 15.7545 & -51.9716 \end{bmatrix}$$

with $\beta = 1.0363$.

5 CONCLUDING REMARKS

The main contribution of this paper resides in the use of a local polytopic representation of the saturation nonlinearity for studying the multiobjective problem of both local stabilization and performance requirements satisfaction with respect to a linear system with saturating controls. Thanks to this representation and the use of relaxation schemes, numerical efficient techniques based on an LMI-framework are proposed in order to compute a effectively saturating state feedback and output observer-based control laws that solves the problem.

The conservativity of the proposed approach is mainly due to the modeling of the closed-loop system by a differential inclusion and the use of a quadratic framework. However, it should be pointed out that this allows to consider the nonlinear behavior of the closed loop system and it is much less conservative than solutions that considers saturation avoidance, as shown in the numerical example. The attempt to find less conservative representations allowing to handle the problem analytical or numerically remains a challenge for future works. It is worth to remark that the closed loop system is in fact piecewise linear. The use of this kind of modeling (Gomes da Silva Jr. and Tarbouriech, 1999b)(Gomes da Silva Jr. and Tarbouriech, 1999a)(Johansson, 1999) has been successfully applied in the analysis context, i.e., for determining estimates for the region of attraction of the origin. However, the design based on this approach is almost impossible to be handled numerically. This comes from the fact that the boundary of the regions that define the piecewise linear models are defined from the gain matrix to be computed.

Since efficient algorithms and software to solve LMI-based problems are available, we can conclude that the proposed approach represents an interesting and easy implementable way to compute saturating control laws. Moreover, the proposed LMI-framework can be easily extended

- to treat uncertain systems, which is not in general possible considering semiglobal approach (Lin and Saberi, 1993);
- to incorporate state and actuator rate constraints;
- to design or specify actuators (considering a given control law satisfying some control requirements, this can be accomplished, by formulating optimization problems having the bound ρ as a variable);
- to generate piecewise linear control laws considering switching surfaces that ensures the asymptotically stability (see (Gomes da Silva Jr and Tarbouriech, 2001)).

Finally, we can say that the discrete-time counterpart of the presented results can be found in (Gomes da Silva Jr and Tarbouriech, 2001).

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